Integrals of simple functions

C is used for an arbitrary constant of integration that can only be determined if something about the value of the integral at some point is known. Thus each function has an infinite number of antiderivatives.

These formulas only state in another form the assertions in the table of derivatives.

Integrals with a singularity

When there is a singularity in the function being integrated such that the integral becomes undefined, it is not Lebesgue integrable, then *C* does not need to be the same on both sides of the singularity. The forms below normally assume the Cauchy principal value around a singularity in the value of *C* but this is ot in general necessary. For instance in

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

There is a singularity at 0 and the integral becomes infinite there. If the integral above was used to give a definite integral between -1 and 1 the answer would be 0. This however is only the value assuming the Cauchy principal value for the integral around the singularity. If the integration was done in the complex plane the result would depend on the path round the origin, in this case the singularity contributes $-i\pi$ when using a path above the origin and $i\pi$ for a path below the origin. A function on the real line could use a completely different value of *C* on either side of the origin as in:

$$\int \frac{1}{x}\,dx = \ln|x| + \begin{cases} A & \text{if } x > 0; \\ B & \text{if } x < 0. \end{cases}$$

Rational functions

more integrals: List of integrals of rational functions

These rational functions have a non-integrable singularity at 0 for $a \le -1$.

$$\int a\,dx = ax + C$$

$$\int x^a\,dx = \frac{x^{a+1}}{a+1} + C$$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

Exponential functions

more integrals: List of integrals of exponential functions

$$\int e^x\,dx = e^x + C$$

$$\int a^x\,dx = \frac{a^x}{\ln a} + C$$

Logarithms

more integrals: List of integrals of logarithmic functions

$$\int \ln x \, dx = x \ln x - x + C$$

$$\int \log_a x \, dx = x \log_a x - \frac{x}{\ln a} + C$$

]Trigonometric functions

more integrals: List of integrals of trigonometric functions

$$\int \sin x \, dx = -\cos x + C$$

$$\int \cos x \, dx = \sin x + C$$

$$\int \tan x \, dx = -\ln |\cos x| + C = \ln |\sec x| + C$$

$$\int \cot x \, dx = \ln |\sin x| + C$$

$$\int \sec x \, dx = \ln |\sec x + \tan x| + C$$

$$\int \csc x \, dx = \ln |\csc x - \cot x| + C$$

$$\int \sec^2 x \, dx = \tan x + C$$

$$\int \csc^2 x \, dx = -\cot x + C$$

$$\int \sec x \, \tan x \, dx = \sec x + C$$

$$\int \csc x \, \cot x \, dx = -\csc x + C$$

$$\int \sin^2 x \, dx = \frac{1}{2}\left(x - \frac{\sin 2x}{2}\right) + C = \frac{1}{2}(x - \sin x \cos x) + C$$

$$\int \cos^2 x \, dx = \frac{1}{2}\left(x + \frac{\sin 2x}{2}\right) + C = \frac{1}{2}(x + \sin x \cos x) + C$$

$$\int \sec^3 x \, dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln |\sec x + \tan x| + C$$

(see integral of secant cubed)

$$\int \sin^n x \, dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x \, dx$$

$$\int \cos^n x \, dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x \, dx$$

Inverse trigonometric functions

more integrals: *List of integrals of inverse trigonometric functions*

$$\int \arcsin x \, dx = x \ \arcsin x + \sqrt{1-x^2} + C$$

$$\int \arccos x \, dx = x \ \arccos x - \sqrt{1-x^2} + C$$

$$\int \arctan x \, dx = x \ \arctan x - \frac{1}{2} \ln \left| 1 + x^2 \right| + C$$

$$\int \operatorname{arccot} x \, dx = x \ \operatorname{arccot} x + \frac{1}{2} \ln \left| 1 + x^2 \right| + C$$

$$\int \operatorname{arcsec} x \, dx = x \ \operatorname{arcsec} x - \operatorname{arcosh} x + C$$

$$\int \operatorname{arccsc} x \, dx = x \ \operatorname{arccsc} x + \operatorname{arcosh} x + C$$

Hyperbolic functions

more integrals: *List of integrals of hyperbolic functions*

$$\int \sinh x \, dx = \cosh x + C$$

$$\int \cosh x \, dx = \sinh x + C$$

$$\int \tanh x \, dx = \ln |\cosh x| + C$$

$$\int \operatorname{cosech} x \, dx = \ln \left| \tanh \frac{x}{2} \right| + C$$

$$\int \operatorname{sech} x \, dx = \arctan (\sinh x) + C$$

$$\int \coth x \, dx = \ln |\sinh x| + C$$

$$\int \operatorname{sech}^2 x \, dx = \tanh x + C$$

Inverse hyperbolic functions

more integrals: *List of integrals of inverse hyperbolic functions*

$$\int \operatorname{arsinh} x \, dx = x \operatorname{arsinh} x - \sqrt{x^2+1} + C$$

$$\int \operatorname{arcosh} x \, dx = x \operatorname{arcosh} x - \sqrt{x^2-1} + C$$

$$\int \operatorname{artanh} x \, dx = x \operatorname{artanh} x + \frac{1}{2} \ln \left(1 - x^2 \right) + C$$

$$\int \operatorname{arcsch} x \, dx = x \operatorname{arcsch} x + \ln \left[x \left(\sqrt{1 + \frac{1}{x^2}} + 1 \right) \right] + C$$

$$\int \operatorname{arsech} x \, dx = x \operatorname{arsech} x - \arctan \left(\frac{x}{x-1} \sqrt{\frac{1-x}{1+x}} \right) + C$$

$$\int \operatorname{arcoth} x \, dx = x \operatorname{arcoth} x + \frac{1}{2} \ln (x^2 - 1) + C$$

Composed functions

$$\int \cos ax \, e^{bx} \, dx = \frac{e^{bx}}{a^2 + b^2} (a \sin ax + b \cos ax) + C$$

$$\int \sin ax \, e^{bx} \, dx = \frac{e^{bx}}{a^2 + b^2} (b \sin ax - a \cos ax) + C$$

$$\int \cos ax \, \cosh bx \, dx = \frac{1}{a^2 + b^2} (a \sin ax \, \cosh bx + b \cos ax \, \sinh bx) + C$$

$$\int \sin ax \, \cosh bx \, dx = \frac{1}{a^2 + b^2} (b \sin ax \, \sinh bx - a \cos ax \, \cosh bx) + C$$

Absolute value functions

$$\int |(ax + b)^n| \, dx = \frac{(ax+b)^{n+2}}{a(n+1)\,|ax+b|} + C \; [\, n \text{ is odd, and } n \neq -1 \,]$$

$$\int |\sin ax| \, dx = \frac{-1}{a} |\sin ax| \cot ax + C$$

$$\int |\cos ax| \, dx = \frac{1}{a} |\cos ax| \tan ax + C$$

$$\int |\tan ax| \, dx = \frac{\tan(ax)[-\ln |\cos ax|]}{a\,|\tan ax|} + C$$

$$\int |\csc ax| \, dx = \frac{-\ln |\csc ax + \cot ax| \sin ax}{a\,|\sin ax|} + C$$

$$\int |\sec ax| \, dx = \frac{\ln |\sec ax + \tan ax| \cos ax}{a\,|\cos ax|} + C$$

$$\int |\cot ax| \, dx = \frac{\tan(ax)[\ln |\sin ax|]}{a\,|\tan ax|} + C$$

Special functions

Ci, Si: Trigonometric integrals, Ei: Exponential integral, li: Logarithmic integral function, erf: Error function

$$\int \operatorname{Ci}(x)dx = x\operatorname{Ci}(x) - \sin x$$

$$\int \operatorname{Si}(x)dx = x\operatorname{Si}(x) + \cos x$$

$$\int \operatorname{Ei}(x)dx = x\operatorname{Ei}(x) - e^x$$

$$\int \operatorname{li}(x)dx = x\operatorname{li}(x) - \operatorname{Ei}(2\ln x)$$

$$\int \frac{\operatorname{li}(x)}{x}\,dx = \ln x\operatorname{li}(x) - x$$

$$\int \operatorname{erf}(x)\,dx = \frac{e^{-x^2}}{\sqrt{\pi}} + x\operatorname{erf}(x)$$

Special functions

Ci, Si: Trigonometric integrals, Ei: Exponential integral, li: Logarithmic integral function, erf: Error function

$$\int_0^\infty \sqrt{x}\,e^{-x}\,dx = \frac{1}{2}\sqrt{\pi}$$ (see also Gamma function)

$$\int_0^\infty e^{-ax^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}$$ (the Gaussian integral)

$$\int_0^\infty x^2 e^{-ax^2}\,dx = \frac{1}{4}\sqrt{\frac{\pi}{a^3}}$$ when a > 0

$$\int_0^\infty x^{2n} e^{-ax^2}\,dx = \frac{2n-1}{2a}\int_0^\infty x^{2(n-1)}e^{-ax^2}\,dx = \frac{(2n-1)!!}{2^{n+1}}\sqrt{\frac{\pi}{a^{2n+1}}} = \frac{(2n)!}{n!2^{2n+1}}\sqrt{\frac{\pi}{a^{2n+1}}}$$

when a > 0, n is 1,2,3,... and !! is the double factorial.

$$\int_0^\infty x^3 e^{-ax^2}\,dx = \frac{1}{2a^2}$$ when a > 0

$$\int_0^\infty x^{2n+1}e^{-ax^2}\,dx = \frac{n}{a}\int_0^\infty x^{2n-1}e^{-ax^2}\,dx = \frac{n!}{2a^{n+1}}$$ when a > 0, n is 0, 1, 2, ...

$$\int_0^\infty \frac{x}{e^x - 1}\,dx = \frac{\pi^2}{6}$$ (see also Bernoulli number)

$$\int_0^\infty \frac{x^3}{e^x - 1}\,dx = \frac{\pi^4}{15}$$

$\int_0^\infty \frac{\sin(x)}{x}\,dx = \frac{\pi}{2}$ (see sinc function and Sine integral)

$\int_0^{\frac{\pi}{2}} \sin^n x\,dx = \int_0^{\frac{\pi}{2}} \cos^n x\,dx = \frac{1\cdot 3\cdot 5\cdot \cdots \cdot (n-1)}{2\cdot 4\cdot 6\cdot \cdots \cdot n}\frac{\pi}{2}$ (if n is an even integer and $n\geq 2$)

$\int_0^{\frac{\pi}{2}} \sin^n x\,dx = \int_0^{\frac{\pi}{2}} \cos^n x\,dx = \frac{2\cdot 4\cdot 6\cdot \cdots \cdot (n-1)}{3\cdot 5\cdot 7\cdot \cdots \cdot n}$ (if n is an odd integer and $n\geq 3$)

$\int_{-\pi}^{\pi} \cos(\alpha x)\cos^n(\beta x)dx = \begin{cases} \frac{2\pi}{2^n}\binom{n}{m} & |\alpha| = |\beta(2m-n)| \\ 0 & \text{otherwise} \end{cases}$ (for α,β,m,n integers with $\beta\neq 0$ and $m,n\geq 0$, see also Binomial coefficient)

$\int_0^1 x^{m-1}(1-x)^{n-1}dx = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$ (the Beta Function)

$\int_{-\pi}^{\pi} \sin(\alpha x)\cos^n(\beta x)dx = 0$ (for α,β real and n non-negative integer, see also Symmetry)

$\int_{-\pi}^{\pi} \sin(\alpha x)\sin^n(\beta x)dx = \begin{cases} (-1)^{(n+1)/2}(-1)^m\frac{2\pi}{2^n}\binom{n}{m} & n \text{ odd},\ \alpha = \beta(2m-n) \\ 0 & \text{otherwise} \end{cases}$

(for α,β,m,n integers with $\beta\neq 0$ and $m,n\geq 0$, see also Binomial coefficient)

$\int_{-\pi}^{\pi} \cos(\alpha x)\sin^n(\beta x)dx = \begin{cases} (-1)^{n/2}(-1)^m\frac{2\pi}{2^n}\binom{n}{m} & n \text{ even},\ |\alpha| = |\beta(2m-n)| \\ 0 & \text{otherwise} \end{cases}$

(for α,β,m,n integers with $\beta\neq 0$ and $m,n\geq 0$, see also Binomial coefficient)

$\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \frac{\pi}{2}$

$\int_0^\infty x^{z-1}\,e^{-x}\,dx = \Gamma(z)$ (where $\Gamma(z)$ is the Gamma function)

$\int_{-\infty}^{\infty} e^{-(ax^2+bx+c)}\,dx = \sqrt{\frac{\pi}{a}}\exp\left[\frac{b^2-4ac}{4a}\right]$ (where $\exp[u]$ is the exponential function e^u, and $a > 0$)

$\int_0^{2\pi} e^{x\cos\theta}d\theta = 2\pi I_0(x)$ (where $I_0(x)$ is the modified Bessel function of the first kind)

$\int_0^{2\pi} e^{x\cos\theta+y\sin\theta}d\theta = 2\pi I_0\left(\sqrt{x^2+y^2}\right)$

$\int_{-\infty}^{\infty} (1+x^2/\nu)^{-(\nu+1)/2}dx = \frac{\sqrt{\nu\pi}\ \Gamma(\nu/2)}{\Gamma((\nu+1)/2))}$, $\nu > 0$, this is related to the probability density function of the Student's t-distribution)

The method of exhaustion provides a formula for the general case when no antiderivative exists:

$$\int_a^b f(x)\,dx = (b-a)\sum_{n=1}^{\infty}\sum_{m=1}^{2^n-1}(-1)^{m+1}2^{-n}f(a+m(b-a)2^{-n}).$$

$$\int_0^1 [\ln(1/x)]^p\,dx = p!$$

The "sophomore's dream"

$$\int_0^1 x^{-x}\,dx = \sum_{n=1}^{\infty} n^{-n} \qquad (= 1.29128599706266\ldots)$$

$$\int_0^1 x^{x}\,dx = -\sum_{n=1}^{\infty}(-1)^n n^{-n} \qquad (= 0.783430510712\ldots)$$